An Extension of the Table of the Toronto Function

By A. H. Heatley

1. Introduction. Twenty years ago I published [4] a very short five-decimal table of the confluent hypergeometric function,[1] M:

$$M(\alpha, \gamma, x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots, \tag{1}$$

and a short table of the Toronto function, T, which is defined in terms of the M-function:

$$T(m, n, r) = r^{2n-m+1}e^{-r^2}\Gamma\left(\frac{m+1}{2}\right)M\left(\frac{m+1}{2}, n+1, r^2\right) \Big/ \Gamma(n+1). \tag{2}$$

Using an IBM 1620 computer the table of the M-function has been extended, for the same parameters as before, from $x = 4$ to $x = 16$ (Table I), and certain gaps in the 1943 table of the T-function have been filled (Table II). The latter function has also been computed for those values of m and n of interest to Wagner [10] and to Carlson [2, Section 3, Tables III and IV]. All these computations were carried to at least twelve significant figures. Values of the function for the new parameters are tabulated to eight significant figures to anticipate other uses.

My former or present tables appear not to include any values published in existing tables of the M-function, except eight values by MacDonald [6], and four values by the Numerical Computation Bureau of Tokyo [8] in their table of the Whittaker function $M_{k,m}(x)$, related to the M-function of this paper by the equation

$$M_{k,m}(x) = x^{m+(1/2)}e^{-x/2}M(m + \tfrac{1}{2} - k, 2m+1, x). \tag{3}$$

2. Interpolation of M. The function $e^{-x}M(\alpha, \gamma, x)$ tabulated in Table I is more suitable for tabulation for these values of α and γ, with $x = 0$ to $x = 16$, than $M(\alpha, \gamma, x)$ itself, but the range of values is still greater than is desirable for interpolation. The function $e^{-\alpha x/\gamma}M(\alpha, \gamma, x)$, proposed by MacDonald [7], is also more suitable than $M(\alpha, \gamma, x)$ throughout this region, but suffers from the same defect. Preliminary studies indicate that $e^{-\alpha x/\gamma}M(\alpha, \gamma, x)$ is more suitable than $e^{-x}M(\alpha, \gamma, x)$ at some values of the parameters and argument, and is less suitable at other values. For the function $M(n/2, \frac{1}{2}, x)$ Horenstein [5] found it desirable, for ease of interpolation, to tabulate different functions of M for different regions of n and x. The same device is probably desirable for regions other than those studied by Horenstein. The problem warrants further study.

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[1] This is the notation of Airey [1, p. 276] and of Fletcher *et al.* [3, v. 1, p. 535]. Other notations are noted by Slater [9, p. 2].

TABLE I
The Confluent Hypergeometric Function
Values tabulated are $10^p \cdot e^{-x} M(\alpha, \gamma, x)$.

x \ α, γ	2.25, 3	p	1.25, 2	p	0.25, 1	p	0.25, 2	p	0.25, 3	p
4.00	4.89001	1	3.64968	1	1.27598	1	4.84744	2	3.27926	2
4.84	4.42262	1	3.21500	1	1.03283	1	3.05444	2	1.77322	2
5.76	4.01038	1	2.85188	1	8.62426	2	1.99274	2	9.61643	3
6.76	3.64846	1	2.54787	1	7.38994	2	1.36034	2	5.35303	3
7.84	3.33116	1	2.29192	1	6.45923	2	9.72571	3	3.11705	3
9.00	3.05275	1	2.07484	1	5.73002	2	7.23899	3	1.91615	3
10.24	2.80792	1	1.88925	1	5.14035	2	5.56307	3	1.24167	3
11.56	2.59194	1	1.72932	1	4.65190	2	4.38133	3	8.41375	4
12.96	2.40072	1	1.59045	1	4.23991	2	3.51696	3	5.90626	4
14.44	2.23075	1	1.46903	1	3.88758	2	2.86663	3	4.26196	4
16.00	2.07909	1	1.36217	1	3.58292	2	2.36652	3	3.14399	4
x \ α, γ	2.75, 3	p	1.75, 2	p	0.75, 1	p	0.75, 2	p	0.75, 3	p
4.00	7.95044	1	7.31570	1	5.87944	1	1.57066	1	7.89533	2
4.84	7.70287	1	7.04009	1	5.58770	1	1.23054	1	5.28481	2
5.76	7.46873	1	6.78720	1	5.33537	1	9.79899	2	3.57985	2
6.76	7.24855	1	6.55556	1	5.11510	1	7.93723	2	2.47145	2
7.84	7.04215	1	6.34332	1	4.92084	1	6.53400	2	1.74535	2
9.00	6.84898	1	6.14849	1	4.74781	1	5.45750	2	1.26167	2
10.24	6.66827	1	5.96920	1	4.59232	1	4.61657	2	9.32367	3
11.56	6.49915	1	5.80373	1	4.45151	1	3.94850	2	7.02845	3
12.96	6.34072	1	5.65056	1	4.32316	1	3.40968	2	5.39205	3
14.44	6.19213	1	5.50834	1	4.20549	1	2.96932	2	4.20102	3
16.00	6.05255	1	5.37591	1	4.09706	1	2.60521	2	3.31802	3

3. Applications of T. One example of the Toronto function has appeared in a study by Wagner on the theory of turbulence and the frequency distribution of wind strength and direction. I have recomputed the two functions tabulated by Wagner to four significant figures [10, p. 399, 400]. Up to $r = 4.0$ the T-function was computed from the M-function; from $r = 3.0$ to $r = 5.0$, from the asymptotic series derived from equation (42) [4]. At and above $r = 3.3$ the asymptotic series agrees to eight significant figures with the result from the M-function.

The results are reported in Table III. Note that

$$2r\pi^{-1/2}T(2, 0, r) = e^{-r^2}M(\tfrac{3}{2}, 1, r^2). \tag{4}$$

For r equal to or greater than 3.0 the asymptotic series gives T to six significant figures with five terms or less.

The report on these functions of Wagner by Fletcher *et al.* [3, v. 1, p. 480] is in error by a factor of two.

In studies on the potential of an ellipsoidal charge distribution, Carlson [2] has used the group of Toronto functions $T(2n + 1, 2n + \frac{1}{2}, r)$, where $n = 0, 1, 2, \cdots$; he did not tabulate numerical values. I have computed these functions using the

TABLE II

The Toronto Function

Values tabulated are $10^p \cdot T(m, n, r)$.

r \ m, n	$-0.5, -2$	p	$-0.5, -1$	p	$-0.5, 0$	p	$-0.5, 1$	p	$-0.5, 2$	p
2.0	7.83531	1	9.35667	1	1.30849	0	1.98838	0	2.69024	0
2.2	8.17550	1	9.50902	1	1.22193	0	1.74901	0	2.45719	0
2.4	8.44702	1	9.61101	1	1.16257	0	1.54729	0	2.15044	0
2.6	8.66504	1	9.68186	1	1.12326	0	1.39777	0	1.85911	0
2.8	8.84164	1	9.73325	1	1.09723	0	1.29526	0	1.62729	0
3.0	8.98616	1	9.77209	1	1.07949	0	1.22739	0	1.46200	0
3.2	9.10565	1	9.80247	1	1.06684	0	1.18228	0	1.35108	0
3.4	9.20545	1	9.82685	1	1.05738	0	1.15123	0	1.27784	0
3.6	9.28958	1	9.84682	1	1.05000	0	1.12878	0	1.22836	0
3.8	9.36113	1	9.86343	1	1.04408	0	1.11172	0	1.19336	0
4.0	9.42246	1	9.87742	1	1.03922	0	1.09825	0	1.16725	0

r \ m, n	$0.5, -2$	p	$0.5, -1$	p	$0.5, 0$	p	$0.5, 1$	p	$0.5, 2$	p
2.0	9.04190	1	9.50859	1	1.01891	0	1.08878	0	1.09461	0
2.2	9.18791	1	9.59698	1	1.01561	0	1.08252	0	1.12509	0
2.4	9.30477	1	9.66365	1	1.01287	0	1.07150	0	1.12738	0
2.6	9.39920	1	9.71498	1	1.01071	0	1.06020	0	1.11580	0
2.8	9.47627	1	9.75530	1	1.00903	0	1.05041	0	1.09988	0
3.0	9.53981	1	9.78756	1	1.00771	0	1.04251	0	1.08454	0
3.2	9.59271	1	9.81378	1	1.00668	0	1.03628	0	1.07156	0
3.4	9.63716	1	9.83539	1	1.00584	0	1.03137	0	1.06113	0
3.6	9.67483	1	9.85343	1	1.00516	0	1.02743	0	1.05286	0
3.8	9.70700	1	9.86864	1	1.00460	0	1.02423	0	1.04625	0
4.0	9.73467	1	9.88159	1	1.00412	0	1.02159	0	1.04089	0

TABLE III

The Toronto Function

r	$2r\pi^{-1/2}\,T(2,0,r)$	$T(2,0,r)$	r	$2r\pi^{-1/2}\,T(2,0,r)$	$T(2,0,r)$
0.0	1.000 0000	infinity	2.5	2.936 4376	1.040 9400
.1	1.004 9938	8.906 5253	2.6	3.044 6035	1.037 7729
.2	1.019 9007	4.519 3171	2.7	3.153 1460	1.034 9640
.3	1.044 5012	3.085 5504	2.8	3.262 0195	1.032 4605
.4	1.078 4416	2.389 3600	2.9	3.371 1857	1.030 2191
.5	1.121 2504	1.987 3645	3.0	3.480 6122	1.028 2041
.6	1.172 3600	1.731 6283	3.1	3.590 2714	1.026 3855
.7	1.231 1315	1.558 6598	3.2	3.700 1397	1.024 7386
.8	1.296 8802	1.436 6627	3.3	3.810 1965	1.023 2420
.9	1.368 9010	1.347 9522	3.4	3.920 4240	1.021 8780
1.0	1.446 4913	1.281 9196	3.5	4.030 8067	1.020 6313
1.1	1.528 9710	1.231 8321	3.6	4.141 3309	1.019 4886
1.2	1.615 6983	1.193 2294	3.7	4.251 9845	1.018 4387
1.3	1.706 0811	1.163 0577	3.8	4.362 7567	1.017 4717
1.4	1.799 5845	1.139 1716	3.9	4.473 6382	1.016 5791
1.5	1.895 7339	1.120 0336	4.0	4.584 6202	1.015 7535
1.6	1.994 1154	1.104 5242	4.1	4.695 6952	1.014 9882
1.7	2.094 3730	1.091 8175	4.2	4.806 8562	1.014 2775
1.8	2.196 2048	1.081 2977	4.3	4.918 0971	1.013 6163
1.9	2.299 3571	1.072 5011	4.4	5.029 4123	1.013 0001
2.0	2.403 6188	1.065 0758	4.5	5.140 7965	1.012 4250
2.1	2.508 8154	1.058 7523	4.6	5.252 2453	1.011 8872
2.2	2.614 8035	1.053 3224	4.7	5.363 7543	1.011 3837
2.3	2.721 4649	1.048 6241	4.8	5.475 3198	1.010 9116
2.4	2.828 7028	1.044 5302	4.9	5.586 9380	1.010 4684
2.5	2.936 4376	1.040 9400	5.0	5.698 6059	1.010 0516

Table IV

The Toronto Function

Values tabulated are $10^p \cdot T(m, n, r)$.

r \ m, n	3.0, 2.5	p	5.0, 4.5	p	7.0, 6.5	p	9.0, 8.5	p
0.0	0.000 0000	0	0.000 0000	0	0.000 0000	0	0.000 0000	0
.1	2.996 1511	4	3.803 6451	7	3.191 4818	10	2.002 3573	13
.2	2.366 3971	3	1.200 7157	5	4.028 3922	8	1.010 7592	10
.3	7.818 6499	3	8.914 1503	5	6.724 7839	7	3.795 0423	9
.4	1.799 3897	2	3.639 9874	4	4.877 2352	6	4.890 5211	8
.5	3.384 8348	2	1.067 0583	3	2.231 1957	5	3.493 1774	7
.6	5.589 7625	2	2.528 9086	3	7.602 1812	5	1.712 2542	6
.7	8.420 2852	2	5.163 1412	3	2.108 2333	4	6.455 2692	6
.8	1.184 0018	1	9.433 1136	3	5.018 0406	4	2.003 8039	5
.9	1.577 6655	1	1.580 8026	2	1.060 9743	3	5.352 0630	5
1.0	2.013 1085	1	2.471 5250	2	2.040 1761	3	1.267 6960	4
1.1	2.478 7758	1	3.649 6037	2	3.628 8902	3	2.720 9695	4
1.2	2.962 5068	1	5.137 0673	2	6.046 4510	3	5.377 9734	4
1.3	3.452 4680	1	6.941 7213	2	9.529 1883	3	9.909 1249	4
1.4	3.937 9017	1	9.056 5524	2	1.431 3918	2	1.718 5458	3
1.5	4.409 6534	1	1.146 0602	1	2.061 9839	2	2.827 1915	3
1.6	4.860 4649	1	1.412 1050	1	2.863 1124	2	4.439 8083	3
1.7	5.285 0584	1	1.699 6139	1	3.848 2324	2	6.690 7063	3
1.8	5.680 0467	1	2.003 8493	1	5.024 8187	2	9.718 6559	3
1.9	6.043 7211	1	2.319 8449	1	6.393 8785	2	1.365 8904	2
2.0	6.375 7665	1	2.642 7054	1	7.950 0074	2	1.863 4938	2
2.1	6.676 9480	1	2.967 8510	1	9.681 9304	2	2.475 0869	2
2.2	6.948 8041	1	3.291 1951	1	1.157 3423	1	3.208 5176	2
2.3	7.193 3733	1	3.609 2531	1	1.360 4473	1	4.068 6367	2
2.4	7.412 9640	1	3.919 1900	1	1.575 2552	1	5.057 0825	2
2.5	7.609 9769	1	4.218 8152	1	1.799 3866	1	6.172 2852	2
2.6	7.786 7749	1	4.506 5408	1	2.030 4481	1	7.409 6681	2
2.7	7.945 5973	1	4.781 3142	1	2.266 1274	1	8.762 0072	2
2.8	8.088 5078	1	5.042 5383	1	2.504 2661	1	1.021 9899	1
2.9	8.217 3707	1	5.289 9882	1	2.742 9106	1	1.177 2288	1
3.0	8.333 8455	1	5.523 7312	1	2.980 3420	1	1.340 7006	1
3.1	8.439 3937	1	5.744 0547	1	3.215 0885	1	1.511 1282	1
3.2	8.535 2937	1	5.951 4041	1	3.445 9236	1	1.687 2206	1
3.3	8.622 6588	1	6.146 3318	1	3.671 8542	1	1.867 7111	1
3.4	8.702 4564	1	6.329 4559	1	3.892 1017	1	2.051 3884	1
3.5	8.775 5268	1	6.501 4290	1	4.106 0787	1	2.237 1196	1
3.6	8.842 6005	1	6.662 9142	1	4.313 3651	1	2.423 8649	1
3.7	8.904 3134	1	6.814 5684	1	4.513 6832	1	2.610 6878	1
3.8	8.961 2205	1	6.957 0302	1	4.706 8747	1	2.796 7576	1
3.9	9.013 8075	1	7.090 9122	1	4.892 8796	1	2.981 3497	1
4.0	9.062 5003	1	7.216 7963	1	5.071 7172	1	3.163 8416	1
4.1	9.107 6741	1	7.335 2309	1	5.243 4699	1	3.343 7068	1
4.2	9.149 6599	1	7.446 7299	1	5.408 2690	1	3.520 5080	1
4.3	9.188 7507	1	7.551 7732	1	5.566 2830	1	3.693 8885	1
4.4	9.225 2066	1	7.650 8072	1	5.717 7076	1	3.863 5642	1
4.5	9.259 2593	1	7.744 2463	1	5.862 7581	1	4.029 3152	1
4.6	9.291 1153	1	7.832 4745	1	6.001 6624	1	4.190 9780	1
4.7	9.320 9597	1	7.915 8474	1	6.134 6560	1	4.348 4379	1
4.8	9.348 9583	1	7.994 6937	1	6.261 9779	1	4.501 6223	1
4.9	9.375 2603	1	8.069 3176	1	6.383 8670	1	4.650 4944	1
5.0	9.400 0000	1	8.140 0000	1	6.500 5600	1	4.795 0480	1

M-function and the appropriate asymptotic series as above. At and above $r = 3.3$, the asymptotic series agrees to eight significant figures with the result from the M-function. The results are reported in Table IV. Values at $n = 0$ have been omitted, for they are simply the error function

$$T\left(1, \frac{1}{2}, r\right) = 2\pi^{-1/2} \int_0^r e^{-y^2}\, dy. \tag{5}$$

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1. J. R. Airey, "Report of the mathematical tables committee," *Annual Report, British Association for the Advancement of Science*, 1926.
2. B. C. Carlson, "Self-energy and potential of an ellipsoidal charge distribution," *Semi-Annual Summary Research Report in Physics*, Iowa State University, January-June, 1959.
3. A. Fletcher, J. C. P. Miller, L. Rosenhead & L. J. Comrie, *An Index of Mathematical Tables*, Vol. I, II, 2nd ed., published for Scientific Computing Service, Ltd., London by Addison-Wesley, Reading, Mass., 1962. MR **26** #365a, b.
4. A. H. Heatley, "A short table of the Toronto function," *Trans. Roy. Soc. Canada Sect. III*, v. 37, 1943, p. 13–29. MR **5**, 245.
5. William Horenstein, "Tables of the confluent hypergeometric function $F(n/2, 1/2; x)$ and related functions," Nat. Bur. Standards Appl. Math. Ser. No. 3, U. S. Government Printing Office, Washington, 1949.
6. A. D. MacDonald, "Properties of the confluent hypergeometric function," *J. Math. Phys.*, v. 28, 1949, p. 183–191. MR **11**, 246.
7. A. D. MacDonald, "Tables of the function $e^{-\alpha z/\gamma}M(\alpha;\ \gamma;\ z)$," *Canad. J. Res. Sect. A* v. 28, 1950, p. 175–179. MR **12**, 207.
8. *Tables of Whittaker functions*, Rep. No. 8, Numerical Computation Bureau, Tsuneta Yano Memorial Society, Tokyo, 1954, MR **16**, 175.
9. L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge Univ. Press, New York, 1960. MR **21**, 5753.
10. A. Wagner, "Theorie der Boigheit und der Haufigkeitsverteilung von Windstärke und Windrichtung," *Gerlands Beitrage zur Geophysik*, v. 24, 1929–30, p. 386–436.

Tables of Values of σ_{2s} Relating to Weierstrass' Elliptic Function

By Chih-Bing Ling

The coefficient σ_{2s} is defined by the double series

$$\sigma_{2s} = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m + n\omega)^{2s}} \qquad (s \geqq 2), \tag{1}$$

where the prime on the summation sign indicates the omission of simultaneous zeros of m and n; ω being a complex quantity. Such a coefficient occurs in the expansion of Weierstrass' elliptic function. Besides, it occurs also in the expansions of Weierstrass' Sigma and Zeta functions.

In two previous papers [1], [2], the author and his associate evaluated to 16D the Weierstrass' elliptic function at half periods and also the two coefficients σ_4 and σ_6 for the following two cases, namely: (i) when $\omega = ai$, and (ii) when $\omega = \frac{1}{2} + ci$. In the former case the primitive period-parallelogram is a rectangle and in the latter a rhombus. It appears that these two cases are the only cases in which σ_4 and σ_6 are both real.

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